The Hilbert Polynomial of the Irreducible Representation of the Rational Cherednik Algebra of Type A_n in Characteristic $p \nmid n$

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Vector Space

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- $\mathbb{k}[[x_1, x_2, \ldots, x_n]]$
- $\mathbb{k}[\partial_x, x]$
- $\mathrm{Mat}_n(\mathbb{k})$

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Example

The algebra $A = \mathbb{k}[x_1, x_2, \ldots, x_n]$ has a grading given by A_i the subspace of homogeneous degree i polynomials.

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The algebra $A = \Bbbk[x_1, \ldots, x_n]$ has the usual grading by degree. Then $\dim(A_i) = \binom{n+i-1}{i}$, so $h_A(z) = \sum_{i \geq 0} \binom{n+i-1}{i} z^i = \frac{1}{(1-z)^n}$.

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A subrepresentation is a subspace $W \subset V$ which remains closed under the action of $\rho(A)$.

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Example

Let $A = \mathbb{k}[S_n]$ be the group algebra of S_n and $V = \mathbb{C}^n$ be a vector space where S_n acts by permutations. Then $\text{Span}\{(1, 1, 1, \ldots, 1)\}$ is an irreducible subrepresentation.

Differential Operators

Let $V = \mathbb{k}[x]$. The differential operator acts by $\partial_x x^k = kx^{k-1}$. In characteristic 0 we can define the algebra of differential operators as a subalgebra in $End(k[x])$ generated by x and ∂_x.

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But in characteristic p, ∂_x^p acts by 0, this is problematic. So to define $k[x, \partial_x]$, use the fact that $[\partial_x, x] = 1$. So $k[x, \partial_x] = \mathbb{k}\langle x, y\rangle/([y, x] = 1)$.

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- The Dunkl operators D_{y_i}
  - An extension of the partial derivative
  - $D_{y_i} = t\partial_{x_i} - c\sum_{k \neq i} \frac{1-\sigma_{ik}}{x_i - x_k}$
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The relevant cases are $t = 1$ and $t = 0$. We will work with $t = 0$. We need more abstract definition for characteristic p as for differential operators.

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The singular polynomials are those which are in the kernel of all Dunkl operators $D_{y_i - y_j}$ for all i, j.

Baby Verma Modules

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- Ideal of symmetric polynomials is a subrepresentation
- Denote by $N_{t,c}$ the quotient by this subrepresentation, which is the baby Verma module

Contravariant form

The contravariant form $B : S\mathfrak{h} \otimes S\mathfrak{h}^* \to \mathbb{k}$ is defined by $B(1,1) = 1$ and for $y \in \mathfrak{h}$, $x \in \mathfrak{h}^*$, $g \in S\mathfrak{h}$, $f \in S\mathfrak{h}^*$, then $B(yg, f) = B(g, D_y(f))$ and $B(g, xf) = B(D_x(g), f)$.
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- The kernel is a subrepresentation
- Define $L_{t,c} = M_{t,c}/\mathrm{ker}B$
- $L_{t,c} = N_{t,c}/\mathrm{ker}B$
- L is an irreducible representation of $H_{t,c}$

Purpose

To find the Hilbert polynomial of the irreducible quotient $L_{t,c}$ in the polynomial representation of the rational Cherednik algebra of type A_n, when the characteristic $p \nmid n$.

The singular polynomials generate a subrepresentation so we would like to find them and remove them.

Goal

To find the smallest d such that degree d polynomials in the simultaneous kernel of the Dunkl operators $D_{y_i - y_j}$ exist, and find the dimension of this kernel.

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$$t = 1 \implies h_{N_{1,c}(\tau)}(z) = \frac{\left(1 - z^{2p}\right)\left(1 - z^{3p}\right)\cdots\left(1 - z^{np}\right)}{(1 - z)^{n-1}},$$

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They showed that

$$h_{L_{t,c}(\tau)}(z) = \left(\frac{1 - z^p}{1 - z}\right)^{n-1} h\left(z^p\right)$$

for some polynomial h with integer coefficients.

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They also proved that $\ker B$ is a maximal proper submodule of the Verma module $M_{t,c}(\tau)$, and that $L_{t,c}(\tau)$ is irreducible.

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- We computed these singular polynomials
- We conjectured a pattern and looked to prove it

Current Progress for $t = 0$

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The case $t = 1$ and $p|n$ was done by Devadas and Sun.

Progress for $p = 2$ and $t = 0$

For $p = 2$, the following polynomials are singular for distinct i, j, k:

- $x_i^2 + x_i x_j + x_j^2$
- $x_i x_j + x_j x_k + x_k x_i$

Progress for $p|n-1$ and $t=0$

For p odd and distinct i, j, k, l, the following polynomials are singular:

- $(x_j + x_k)(x_i - x_j - x_k)$
- $(x_i - x_j)(x_k - x_l)$

Conjectures

- The Hilbert polynomial for $p = 2$ and $t = 0$ is $h_{L_{0,c}}(z) = 1 + (n-1)z + (n-1)z^2 + z^3$
- Etingof conjectures that for $n = kp + r$, then $h_{L_{0,c}}(z) = [r]_z![p]_z Q_r(n,z)$ and $h_{L_{1,c}}(z) = [r]_{z^p}![p]_{z^p}[p]_z^{n-1} Q_r(n, z^p)$, for $Q_r(n,z) = \binom{n-1}{r-1} z^{r+1} + \sum_{i=0}^{r} \binom{n-r-2+i}{i} z^i$, $[k]_z! = [k]_z[k-1]_z \cdots [1]_z$, and $[w]_z = \frac{1-z^w}{1-z}$

Future Work

In the future, we would like to find the Hilbert polynomials for $L_{t,c}$, and the singular polynomials for various p, n. We would like to study more cases in $t = 0$ and prove irreducibility, then consider the connection between $t = 0$ and $t = 1$.

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